

Logarithmic intertwining operators
and
the space of conformal blocks over the projective line

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Abstract

We show that the space of logarithmic intertwining operators among logarithmic modules for a vertex operator algebra is isomorphic to the space of 3-point conformal blocks over the projective line. This is considered as a generalization of Zhu's result for ordinary intertwining operators among ordinary modules.

1 Introduction

One of the most important problems in representation theory of vertex operator algebras is to determine fusion rules which are the dimensions of intertwining operators among three modules for vertex operator algebras. Intertwining operators of the type $\binom{M^3}{M^1 M^2}$ are linear maps $I(-, z) : M^1 \rightarrow \text{Hom}_{\mathbb{C}}(M^2, M^3)[[z, z^{-1}]]$ with several axioms (see [FHL]) where M^i ($i = 1, 2, 3$) are modules for a vertex operator algebra.

The definition of intertwining operators given in [FHL] treats modules on which L_0 acts as a semisimple operator. However, in general, we have to consider modules which do not decompose into L_0 -eigenspaces but do into generalized L_0 -eigenspaces. Such modules are called *logarithmic modules* in [M1].

A notion of *logarithmic intertwining operators* among logarithmic modules is introduced in [M1]. Logarithmic intertwining operators may involve logarithmic terms. It is shown in [M1] that a logarithmic intertwining operator among ordinary modules is nothing but the so-called intertwining operator. Several examples of logarithmic modules are found and logarithmic intertwining operators among these modules are constructed (see eg. [M1], [M2], [AM]).

On the other hand in conformal field theory its important feature is a notion of conformal blocks associated with vertex operator algebras. Mathematically rigorous formulation of N -point conformal blocks on Riemann surfaces associated with vertex operator algebras is given in [Z1] with the assumption that the corresponding vertex operator algebra is quasi-primary generated. It is shown in [Z1] that the space of 3-point conformal blocks over the projective line \mathbb{P}^1 is

isomorphic to the space of intertwining operators among ordinary modules for a vertex operator algebra.

In this paper we give a sort of generalization of Zhu's result in the case that the modules are logarithmic. More precisely we are going to prove that the space of 3-point conformal blocks over the projective line is isomorphic to the space of logarithmic intertwining operators without the assumption that a vertex operator algebra is quasi-primary generated. Taking the formulation of the space of coinvariants in [NT] we do not have to assume that a vertex operator algebra is quasi-primary generated.

The study on logarithmic intertwining operators is very important since if we could know its dimension from S -matrix obtained by formal characters in fact if a vertex operator algebra is rational and satisfies several conditions the dimension of intertwining operators is completely determined by S -matrix. However it is left for further studies.

The paper is organized as follows. In section 2 we recall the definition of vertex operator algebras and their modules. The definition of logarithmic modules is located here. Also we describe the space of conformal blocks over \mathbb{P}^1 according to [NT].

In section 3 we recall the definition and properties of logarithmic intertwining operators which are given in [M1] and [HLZ].

We state the main theorem in this paper and give a proof in section 4. The linear maps between the space of logarithmic intertwining operators and the space of 3-point conformal blocks are defined and it is proved that these maps are well-defined and mutually inverse.

2 Vertex operator algebras and the space of conformal blocks over the projective line

Throughout this paper we use the notation $\mathbb{N} = \{0, 1, 2, \dots\}$.

2.1 Vertex operator algebras and current Lie algebras

A *vertex operator algebra* is a \mathbb{N} -graded vector space $V = \bigoplus_{k=0}^{\infty} V_k$ with $\dim V_k < \infty$ ($k \in \mathbb{Z}_{\geq 0}$) equipped with a linear map

$$Y(-, z) : V \rightarrow \text{End}(V)[[z, z^{-1}]], \quad Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \quad (2.1)$$

and with distinguished vectors $\mathbf{1} \in V_0$ called the *vacuum vector* and $\omega \in V_2$ called the *Virasoro vector* satisfying the following axioms (see e.g. [FHL], [MN]):

- (1) For any pair of vectors in V there exists a nonnegative integer N such that $a_{(n)}b = 0$ for all integers $n \geq N$.

(2) For any vectors $a, b, c \in V$ and integers $p, q, r \in \mathbb{Z}$,

$$\begin{aligned} \sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)} b)_{(p+q-i)} c \\ = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} (a_{(p+r-i)} b_{(q+i)} c - (-1)^r b_{(q+r-i)} a_{(p+i)} c) \end{aligned} \quad (2.2)$$

hold.

(3) $Y(\mathbf{1}, z) = \text{id}_V$.

(4) $Y(a, z)\mathbf{1} \in V[[z]]$ and $a_{(-1)}\mathbf{1} = a$.

(5) Set $L_n = \omega_{(n+1)}$. Then $\{L_n \mid n \in \mathbb{Z}\}$ together with the identity map on V give a representation of the Virasoro algebra on V with central charge $c_V \in \mathbb{C}$.

(6) $L_0 a = k a$ for any $a \in V_k$ and nonnegative integers k .

(7) $\frac{d}{dz} Y(a, z) = Y(L_{-1} a, z)$ for any $a \in V$.

(8) Denote $|a| = k$ for any $a \in V_k$ and then

$$|a_{(n)} b| = |a| + |b| - 1 - n \quad (2.3)$$

for any homogeneous $b \in V$ and $n \in \mathbb{Z}$.

In order to define the space of conformal blocks we introduce the spaces $V^{(1)} = \bigoplus_{k=0}^{\infty} V_k \otimes \mathbb{C}((\xi))(d\xi)^{1-k}$ and $V^{(0)} = \bigoplus_{k=0}^{\infty} V_k \otimes \mathbb{C}((\xi))(d\xi)^{-k}$. Let $\nabla : V^{(0)} \rightarrow V^{(1)}$ be the linear map defined by

$$v \otimes f(\xi)(d\xi)^{-n} = L_{-1} v \otimes f(\xi)(d\xi)^{-k} + v \otimes \frac{df(\xi)}{d\xi} (d\xi)^{1-k}. \quad (2.4)$$

We set $\mathfrak{g} = V^{(1)}/\nabla V^{(0)}$ and denote the image of $a \otimes f(\xi)(d\xi)^{1-k} \in V_k \otimes \mathbb{C}((\xi))(d\xi)^{1-k}$ by $J(a, f)$. Then we have:

Proposition 2.1.1 ([NT], Proposition 2.1.1). *The vector space \mathfrak{g} is a Lie algebra with the bracket*

$$[J(a, f), J(b, g)] = \sum_{m=0}^{|a|+|b|-1} \frac{1}{m!} J(a_{(m)} b, \frac{d^m f}{d\xi^m} g) \quad (2.5)$$

for homogeneous $a, b \in V$.

The Lie algebra \mathfrak{g} is called the *current Lie algebra*. Let us denote $J_n(a) = J(a, \xi^{n+|a|-1})$.

Applying the construction of the current Lie algebra to the vector space $\bigoplus_{k=0}^{\infty} V_k \otimes \mathbb{C}[\xi, \xi^{-1}](d\xi)^{1-k}$, we have a graded Lie algebra $\bar{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \bar{\mathfrak{g}}_n$ where the vector space $\bar{\mathfrak{g}}_n$ is linearly spanned by $J_n(a)$ ($a \in V$). The Lie algebra $\bar{\mathfrak{g}}$ is a Lie subalgebra of \mathfrak{g} . The following proposition plays an important role when we define duality functor on the category of V -modules.

Proposition 2.1.2 ([NT], Proposition 4.1.1). *The linear map $\theta : \bar{\mathfrak{g}} \rightarrow \bar{\mathfrak{g}}$ defined by*

$$\theta(J_n(a)) = (-1)^{|a|} \sum_{j=0}^{\infty} \frac{1}{j!} J_{-n}(L_1^j a) \quad (2.6)$$

for $a \in V$ and $n \in \mathbb{Z}$ is an anti-Lie algebra involution.

2.2 Modules for vertex operator algebras

Let M be a weak V -module (see [DLM] for the definition). A weak V -module M is called \mathbb{N} -gradable if it admits a decomposition $M = \bigoplus_{n \in \mathbb{N}} M_{(n)}$ such that

$$a_{(n)} M_{(k)} \subseteq M_{(|a|+k-1-n)} \quad (2.7)$$

for homogeneous $a \in V$ and $n \in \mathbb{Z}$.

Let M be a weak V -module. A weak V -module is called a *logarithmic module* if M decomposes into a direct sum of generalized L_0 -eigenspaces.

Let $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M_{(h+n)}$ be a logarithmic module with a complex number h and

$$M_{(h+n)} = \{x \in M \mid (L_0 - h - n)^{k+1} x = 0 \text{ for a nonnegative integer } k\}. \quad (2.8)$$

Obviously M is a \mathbb{N} -gradable V -module.

In this paper a V -module M is always a logarithmic V -module satisfies the following conditions.

- i) There exist a complex number h and a nonnegative integer k such that $M = \bigoplus_{n=0}^{\infty} M_{(h+n)}$ with $M_{(h+n)} = \{u \in M \mid (L_0 - h - n)^{k+1} u = 0\}$ for all $n \in \mathbb{Z}$.
- ii) $\dim M_{(h+n)} < \infty$ for all nonnegative integers n . We denote $|u| = h + n$ for $u \in M_{(h+n)}$ for short.

We remark that any V -module in this paper is a V -module in the sense of [NT] and [MNT].

Let k be a nonnegative integer and let \mathcal{C}_k be the category consisting of V -modules whose homogeneous subspaces are annihilated by $(L_0 - h - n)^{k+1}$. Then it follows that $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_k \subseteq \cdots$.

Any V -module M is a $\bar{\mathfrak{g}}$ -module by the action

$$J_n(a)u = a_{(|a|-1+n)}u \quad (2.9)$$

for any homogeneous $a \in V$ and $u \in M$ (cf. [DLM], [NT]). For any $a \in V$ and $u \in M$, there exists a nonnegative integer n_0 such that $a_{(n)}u = 0$ for all $n \geq n_0$. Therefore, the V -module M is also a \mathfrak{g} -module by the action (2.9).

Let us denote the restricted dual of a V -module $M = \bigoplus_{n=0}^{\infty} M_{(h+n)}$ by $D(M) = \bigoplus_{n=0}^{\infty} M_{(h+n)}^*$ where $M_{(h+n)}^* = \text{Hom}_{\mathbb{C}}(M_{(h+n)}, \mathbb{C})$. A $\bar{\mathfrak{g}}$ -module structure on $D(M)$ can be defined by letting

$$\langle J_n(a)\varphi, u \rangle = \langle \varphi, \theta(J_n(a))u \rangle \quad (2.10)$$

for all $\varphi \in D(M)$ and $u \in M$. The following proposition is known.

Proposition 2.2.1 ([NT], Proposition 4.2.1, cf. [FHL], Theorem 5.2.1). *There exists a unique V -module structure on $D(M)$ which extends \mathfrak{g} -module structure.*

Since $\langle L_n \varphi, u \rangle = \langle \varphi, L_{-n} u \rangle$ for all $\varphi \in D(M)$ and $u \in M$, we see that $D(M) = \bigoplus_{n=0}^{\infty} D(M)_{(h+n)}$ and $D(M) \in \mathcal{C}_k$ for any $M \in \mathcal{C}_k$.

2.3 The space of conformal blocks over the projective line

Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the projective line and z its inhomogeneous coordinate. Let $A = \{1, 2, \dots, N, \infty\}$ and let us fix a set $p_A = (p_a)_{a \in A}$ of $N + 1$ distinct points $p_a \in \mathbb{P}^1$ ($a \in A$) with $p_{\infty} = \infty$. We write $\xi_a = z - p_a$ ($a \neq \infty$) and $\xi_{\infty} = z$, respectively.

We denote by $H^0(\mathbb{P}^1, \Omega^{1-k}(*p_A))$ the vector space of global meromorphic $(1-k)$ -differentials whose poles are located only at p_a ($a \in A$). Set $H(V, *p_A)^{(1)} = \bigoplus_{k=0}^{\infty} V_k \otimes H^0(\mathbb{P}^1, \Omega^{1-k}(*p_A))$ and $H(V, *p_A)^{(0)} = \bigoplus_{k=0}^{\infty} V_k \otimes H^0(\mathbb{P}^1, \Omega^{-k}(*p_A))$. Define the linear map $\nabla : H(V, *p_A)^{(0)} \rightarrow H(V, *p_A)^{(1)}$ by

$$a \otimes f(z)(dz)^{1-k} \mapsto L_{-1}a \otimes f(z)(dz)^{-k} + a \otimes \frac{df(z)}{dz}(dz)^{1-k} \quad (a \in V_k). \quad (2.11)$$

We set

$$\mathfrak{g}(\mathbb{P}^1, *p_A) = H(V, *p_A)^{(1)} / \nabla H(V, *p_A)^{(0)}. \quad (2.12)$$

It is shown (cf. [NT, Proposition 5.1.1]) that the vector space $\mathfrak{g}(\mathbb{P}^1, *p_A)$ is a Lie algebra with the bracket

$$\begin{aligned} & [a \otimes f(z)(dz)^{1-|a|}, b \otimes g(z)(dz)^{1-|b|}] \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} a_{(m)} b \otimes \frac{d^m f(z)}{dz^m} g(z)(dz)^{2-|a|-|b|+m}. \end{aligned} \quad (2.13)$$

For each $a \in A$ we define the linear map

$$i_a : H^0(\mathbb{P}^1, \Omega^k(*p_A)) \rightarrow \begin{cases} \mathbb{C}((\xi_a))(d\xi_a)^k, & a \in A \setminus \{\infty\} \\ \mathbb{C}((\xi_{\infty}^{-1}))(d\xi_{\infty})^k, & a = \infty \end{cases} \quad (2.14)$$

by taking the Laurent expansion at $z = p_a$ in terms of the coordinate ξ_a . We denote $i_a f(z)(dz)^k$ by $f_a(\xi_a)(d\xi_a)^k$.

For any $a \in A \setminus \{\infty\}$, we define the linear map $j_a : \mathfrak{g}(\mathbb{P}^1, *p_A) \rightarrow \mathfrak{g}$ by $j_a(a \otimes f(z)(dz)^{1-k}) = a \otimes f_a(\xi_a)(d\xi_a)^{1-k}$ and the linear map $j_{\infty} : \mathfrak{g}(\mathbb{P}^1, *p_A) \rightarrow \mathfrak{g}$ by $j_{\infty}(a \otimes f(z)(dz)^{1-k}) = -\theta(a \otimes f_{\infty}(\xi_{\infty})(d\xi_{\infty})^{1-k})$. Then the linear map j_{∞} is well-defined since

$$j_{\infty}(a \otimes f(z)(dz)^{1-k}) = - \sum_{n \leq n_0} f_n \theta(J_n(a)), \quad \theta(J_n(a)) = (-1)^k J_{-n}(e^{L_1} a), \quad (2.15)$$

where $f_{\infty}(\xi_{\infty}) = \sum_{n \leq n_0} f_n \xi_{\infty}^{n+k-1}$ (see [NT]). The following proposition is fundamental.

Proposition 2.3.1 ([NT], Proposition 5.1.3). *For any $a \in A$, the linear map $j_a : \mathfrak{g}(\mathbb{P}^1, *p_A) \rightarrow \mathfrak{g}$ is a Lie algebra homomorphism.*

Let M^a ($a \in A$) be V -modules. We set $M_A = \bigotimes_{a \in A} M^a$ and $\mathfrak{g}_A = \mathfrak{g}^{\oplus |A|}$. Let $\rho_a : \mathfrak{g} \rightarrow \text{End}(M^a)$ be the representation defined by (2.9) for $a \in A$. Then the linear map $\rho_A : \mathfrak{g}_A \rightarrow \text{End}(M_A)$ defined by $\rho_A = \bigoplus_{a \in A} \rho_a$ is a representation of the Lie algebra \mathfrak{g}_A on M_A . We denote the image of the Lie algebra homomorphism $j_A = \sum_{a \in A} j_a$ by $\mathfrak{g}_{p_A}^{\text{out}}$, which acts on M_A via ρ_A . The following definition is given by [NT].

Definition 2.3.2. The vector space $C^*(M_A, p_A) = \text{Hom}_{\mathbb{C}}(M_A / \mathfrak{g}_{p_A}^{\text{out}} M_A, \mathbb{C})$ is called the space of conformal blocks at p_A .

3 Logarithmic intertwining operators

In this section, we recall the notion of logarithmic intertwining operators and their properties according to [M1].

3.1 Definition

Definition 3.1.1 ([M1]). Let M^1 , M^2 and M^3 be weak V -modules. A *logarithmic intertwining operator of the type* $\left(\begin{smallmatrix} M^3 \\ M^1 \ M^2 \end{smallmatrix} \right)$ is a linear map

$$I(-, z) : M^1 \rightarrow \text{Hom}_{\mathbb{C}}(M^2, M^3)\{z\}[\log z] \quad (3.1)$$

$$I(u, z) = \sum_{n=0}^d \sum_{\alpha \in \mathbb{C}} u_{(\alpha)}^n z^{-\alpha-1} (\log z)^k \quad (3.2)$$

with the following properties:

i) (Truncation condition) For any $u_1 \in M^1$, $u_2 \in M^2$ and $0 \leq k \leq d$,

$$(u_1)_{(\alpha)}^k u_2 = 0 \quad (3.3)$$

for sufficiently large $\text{Re}(\alpha)$.

ii) (L_{-1} -derivative property) For any $u_1 \in M^1$,

$$I(L_{-1}u_1, z) = \frac{d}{dz} I(u_1, z). \quad (3.4)$$

iii) For all $a \in V$, $u_1 \in M_1$, $u_2 \in M_2$, $\alpha \in \mathbb{C}$, $0 \leq n \leq d$ and $p, q \in \mathbb{Z}$, we have

$$\begin{aligned} & \sum_{i=0}^{\infty} \binom{p}{i} (a_{(q+i)} u_1)_{(\alpha+p-i)}^n \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{q}{i} (a_{(p+q-i)} (u_1)_{(\alpha+i)}^n - (-1)^q (u_1)_{(\alpha+q-i)}^n a_{(p+i)}). \end{aligned} \quad (3.5)$$

We denote the space of logarithmic intertwining operators of the type $\left(\begin{smallmatrix} M^3 \\ M^1 \ M^2 \end{smallmatrix} \right)$ by $I\left(\begin{smallmatrix} M^3 \\ M^1 \ M^2 \end{smallmatrix} \right)$, that is, we use the same notation as usual intertwining operators.

Setting $q = 0$ and $p = 0$ in (3.5), respectively, we have

$$[a_{(p)}, (u_1)_{(\alpha)}^n] = \sum_{i=0}^{\infty} \binom{p}{i} (a_{(i)} u_1)_{(\alpha+p-i)}^n, \quad (3.6)$$

$$(a_{(q)} u_1)_{(\alpha)}^n = \sum_{i=0}^{\infty} (-1)^i \binom{q}{i} \{a_{(q-i)} (u_1)_{(\alpha+i)}^n - (-1)^q a_{(\alpha+q-i)}^n a_{(i)}\} \quad (3.7)$$

and we call, by abuse of terminologies, the *commutator formula* and *associativity formula*, respectively. By the commutator formula, we have

$$[L_{-1}, u_{(\alpha)}^n] = (L_{-1} u)_{(\alpha)}^n \quad (3.8)$$

for any $u \in M^1$ and $0 \leq n \leq d$. By the associativity formula, (3.8) and L_{-1} -derivative property, we have

$$(L_0 u)_{(\alpha)}^n = \begin{cases} [L_0, (u)_{(\alpha)}^n] + (\alpha + 1)(u)_{(\alpha)}^n - (n + 1)(u)_{(\alpha)}^{n+1} & 0 \leq n \leq d - 1, \\ [L_0, (u)_{(\alpha)}^n] + (\alpha + 1)(u)_{(\alpha)}^n & n = d \end{cases} \quad (3.9)$$

for any $u \in M^1$.

3.2 Properties for logarithmic intertwining operators

Let $M^i = \bigoplus_{n=0}^{\infty} M_{(h_i+n)}^i$ ($i = 1, 2, 3$) be objects in \mathcal{C}_{k_i} for nonnegative integers k_i ($i = 1, 2, 3$) and complex numbers h_i ($i = 1, 2, 3$). Suppose that a logarithmic intertwining operator $I(-, z)$ of the type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$ is of the form

$$I(u_1, z) = \sum_{n=0}^d \sum_{\alpha \in \mathbb{C}} (u_1)_{(\alpha)}^n z^{-\alpha-1} (\log z)^n. \quad (3.10)$$

For any homogeneous element $u_i \in M^i$ ($i = 1, 2$), we introduce notations

$$x_1(u_1)_{(\alpha)}^n u_2 = ((L_0 - |u_1|)u_1)_{(\alpha)}^n u_2, \quad (3.11)$$

$$x_2(u_1)_{(\alpha)}^n u_2 = (u_1)_{(\alpha)}^n (L_0 - |u_2|)u_2, \quad (3.12)$$

$$x_3(u_1)_{(\alpha)}^n u_2 = (L_0 + \alpha + 1 - |u_1| - |u_2|)(u_1)_{(\alpha)}^n u_2. \quad (3.13)$$

Note that these operations x_1 and x_2 are mutually commutative (see [M1]). By using these operations we get:

Lemma 3.2.1 ([HLZ], Lemma 3.8). *Let*

$$I(-, z) = \sum_{n=0}^d \sum_{\alpha \in \mathbb{C}} (-)_{(\alpha)}^n z^{-\alpha-1} (\log z)^n \quad (3.14)$$

be a logarithmic intertwining operator of the type $\left(\begin{smallmatrix} M^3 \\ M^1 M^2 \end{smallmatrix}\right)$ and let p, q be integers such that $p \geq 0$ and $0 \leq q \leq d$. Then

$$x_3^p (u_1)_{(\alpha)}^q u_2 = \sum_{\ell=0}^N \binom{p}{\ell} \frac{(q+\ell)!}{q!} (x_1 + x_2)^{p-\ell} (u_1)_{(\alpha)}^{q+\ell} u_2$$

for homogeneous $u_1 \in M^1$ and $u_2 \in M^2$ where $N = \min\{p, d - q\}$.

The following proposition is proved in [M1] by using differential equations and in [HLZ, Proposition 3.9] by using Lemma 3.2.1.

Proposition 3.2.2 ([M1], Proposition 1.10). *Suppose that $M^i \in \mathcal{C}_{k_i}$ ($i = 1, 2, 3$) for nonnegative integers k_i and that $M^i = \bigoplus_{n=0}^{\infty} M_{(h_i+n)}^i$ for complex numbers h_i ($i = 1, 2, 3$). Let $I(-, z) \in I\left(\begin{smallmatrix} M^3 \\ M^1 M^2 \end{smallmatrix}\right)$ be a logarithmic intertwining operator such that*

$$I(u_1, z) = \sum_{n=0}^d \sum_{\alpha \in \mathbb{C}} (u_1)_{(\alpha)}^n z^{-\alpha-1} (\log z)^n \quad (u_1 \in M^1). \quad (3.15)$$

- (1) *For any homogeneous $u_i \in M^i$ ($i = 1, 2$) we have $|(u_1)_{(\alpha)}^n u_2| = |u_1| + |u_2| - 1 - \alpha$ for all $0 \leq n \leq d$.*
- (2) *For any $u_i \in M^i$ ($i = 1, 2$) we have*

$$I(u_1, z) u_2 \in \sum_{n=0}^{k_1+k_2+k_3} M^3((z)) z^{h_3-h_1-h_1} (\log z)^n.$$

4 The space of 3-point conformal blocks and logarithmic intertwining operators

In this section we focus on 3-point conformal blocks in conformal field theories over the projective line. We prove that the space of 3-point conformal blocks over \mathbb{P}^1 is isomorphic to the space of logarithmic intertwining operators. The almost same result is found in [Z1], however, the categories of modules of us and the one in [Z1] are slightly different.

4.1 Main theorem

Set $A = \{1, 2, \infty\}$ and let $p_A = \{0, 1, \infty\}$ be the set of points on \mathbb{P}^1 . Let z be the inhomogeneous coordinate of \mathbb{P}^1 . The $\xi_0 = z$, $\xi_1 = z - 1$ and $\xi_\infty = z$ are local coordinate of \mathbb{P}^1 at 0, 1, and ∞ , respectively. Take V -modules M^1, M^2 and M^3 . We assume that there exist complex numbers $h_i \in \mathbb{C}$ ($i = 1, 2, 3$) such that $M^i = \bigoplus_{n=0}^{\infty} M_{(h_i+n)}^i$ and that $M^i \in \mathcal{C}_{k_i}$ ($i = 1, 2, 3$) for nonnegative integers k_i ($i = 1, 2, 3$). Let us set $M_A = M^1 \otimes M^2 \otimes M^3$. We denote the space of conformal blocks at $p_A = \{0, 1, \infty\}$ by $C^*(M_A, p_A)$. Then we can now state the main theorem of the paper which is a generalization of Zhu's result [Z1, Proposition 7.4].

Theorem 4.1.1. *Let $M^i (i = 1, 2, 3)$ be V -modules with $M^i = \bigoplus_{n=0}^{\infty} M_{(h_i+n)}^i$ and $M^i \in \mathcal{C}_{k_i} (i = 1, 2, 3)$ for nonnegative integers $k_i (i = 1, 2, 3)$. The space of conformal blocks $C^*(M_A, p_A)$ at $p_A = \{0, 1, \infty\}$ is isomorphic to the space of logarithmic intertwining operators of the type $\binom{D(M^3)}{M^2 \ M^1}$*

Let $C_2(V)$ be the vector subspace of V spanned by vectors of the form $a_{(2)}b (a, b \in V)$. If $\dim V/C_2(V) < \infty$ we say that V satisfies *Zhu's finiteness condition* which is introduced in [Z2].

By combining [NT, Theorem 5.8.1] and the theorem we get:

Corollary 4.1.2. *If V satisfies Zhu's finiteness condition then the space of intertwining operators is finite-dimensional.*

4.2 Proof of Theorem 4.1.1

For any logarithmic intertwining operator $I(-, z)$ of the type $\binom{D(M^3)}{M^2 \ M^1}$, we define $F \in \text{Hom}_{\mathbb{C}}(M_A, \mathbb{C})$ by

$$\langle F, u_1 \otimes u_2 \otimes u_3 \rangle = \langle I(u_2, 1)u_1, u_3 \rangle \quad (4.1)$$

for any $u_1 \in M^1, u_2 \in M^2$ and $u_3 \in M^3$. For any V -module $M \in \mathcal{C}_k$ we define the operator $z^{L_0} : M \rightarrow M\{z\}[\log z]$ by

$$z^{L_0}u = \sum_{j=0}^k \frac{1}{j!} (L_0 - |u|)^j z^{|u|} (\log z)^j. \quad (4.2)$$

For any $x \in C^*(M_A, p_A)$, we define $I_x(-, z) \in \text{Hom}_{\mathbb{C}}(M^1, D(M^3))\{z\}[\log z]$ by

$$\begin{aligned} & \langle I_x(u_2, z)u_1, u_3 \rangle \\ &= \langle x, z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \text{ for all } u_i \in M^i (i = 1, 2, 3). \end{aligned} \quad (4.3)$$

We are going to give a prove of the theorem by dividing its into three steps. In step 1 we prove that F belongs to $C^*(M_A, p_A)$ and show that I_x is a logarithmic intertwining operator among V -modules in step 2. The final step is devoted to the proof that the correspondence between F and I_x is one-to-one.

(Step 1) In order to prove that F belongs to $C^*(M_A, p_A)$, by the definition of the space of conformal blocks, it is sufficient to prove that

$$\begin{aligned} & \left\langle F, j_0(a \otimes f(z)(dz)^{1-k})u_1 \otimes u_2 \otimes u_3 \right\rangle \\ &+ \left\langle F, u_1 \otimes j_1(a \otimes f(z)(dz)^{1-k})u_2 \otimes u_3 \right\rangle \\ &+ \left\langle F, u_1 \otimes u_2 \otimes j_{\infty}(a \otimes f(z)(dz)^{1-k})u_3 \right\rangle = 0 \end{aligned} \quad (4.4)$$

for all $a \in V_k$ and $f(z)(dz)^{1-k} \in H^0(\mathbb{P}^1, \Omega^{1-k}(*p_A))$. It is well known that $\{z^p(z-1)^q(dz)^{1-k} \mid p, q \in \mathbb{Z}\}$ is a topological basis of $H^0(\mathbb{P}^1, \Omega^{1-k}(*p_A))$. Therefore, it is enough to show (4.4) for $f(z) = z^p(z-1)^q$ ($p, q \in \mathbb{Z}$). First of all we have

$$\begin{aligned} j_0(a \otimes z^p(z-1)^q(dz)^{1-k})u_1 &= \left(\sum_{i=0}^{\infty} (-1)^{q-i} \binom{q}{i} a \otimes \xi_0^{p+i} (d\xi_0)^{1-k} \right) u_1 \\ &= \sum_{i=0}^{\infty} (-1)^{q-i} \binom{q}{i} J_{p+i-k+1}(a) u_1 \\ &= \sum_{i=0}^{\infty} (-1)^{q-i} a_{(p+i)} u_1 \end{aligned} \quad (4.5)$$

and secondly

$$\begin{aligned} j_1(a \otimes z^p(z-1)^q(dz)^{1-k})u_2 &= \left(\sum_{i=0}^{\infty} \binom{p}{i} a \otimes \xi_1^{q+i} (d\xi_1)^{1-k} \right) u_2 \\ &= \sum_{i=0}^{\infty} \binom{p}{i} J_{q+i-k+1}(a) u_2 \\ &= \sum_{i=0}^{\infty} \binom{p}{i} a_{(q+i)} u_2 \end{aligned} \quad (4.6)$$

and finally

$$\begin{aligned} j_{\infty}(a \otimes z^p(z-1)^q(dz)^{1-k})u_3 &= -\theta \left(\sum_{i=0}^{\infty} (-1)^i \binom{q}{i} a \otimes \xi_{\infty}^{p+q-i} (d\xi_{\infty})^{1-k} \right) u_3 \\ &= -\sum_{i=0}^{\infty} (-1)^i \binom{q}{i} \theta(J_{p+q-i-k+1}(a)) u_3 \end{aligned} \quad (4.7)$$

for all $a \in V_k$ and $p, q \in \mathbb{Z}$.

By (4.5)–(4.7), the definition of the functional F , Proposition 2.1.2 and Proposition 3.2.2, the left-hand side of (4.4) is equal to

$$\begin{aligned} &\sum_{i=0}^{\infty} (-1)^{q-i} \binom{q}{i} \left\langle (u_2)_{(\alpha+q-i)}^0 a_{(p+i)} u_1, u_3 \right\rangle \\ &\quad + \sum_{i=0}^{\infty} \binom{p}{i} \left\langle (a_{(q+i)} u_2)_{(\alpha+p-i)}^0 u_1, u_3 \right\rangle \\ &\quad - \sum_{i=0}^{\infty} (-1)^i \binom{q}{i} \left\langle a_{(p+q-i)} (u_2)_{(\alpha-i)}^0 u_1, u_3 \right\rangle \end{aligned} \quad (4.8)$$

where $\alpha = |u_1| + |u_2| - |u_3| + k - 2 - p - q$, which vanishes by (3.5). Hence (4.4) is proved.

(Step 2) We now prove that $I_x(-, z) \in I_{\binom{D(M^3)}{M^2 \ M^1}}$.

Since $M^i = \bigoplus_{n=0}^{\infty} M_{(h_i+n)}^i \in \mathcal{C}_{k_i}$ ($i = 1, 2, 3$) we have

$$\begin{aligned} z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 &= \sum_{n_1=0}^{k_1} \sum_{n_2=0}^{k_2} \sum_{n_3=0}^{k_3} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} \\ &\quad \times (L_0 - |u_1|)^{n_1}u_1 \otimes (L_0 - |u_2|)^{n_2}u_2 \otimes (L_0 - |u_3|)^{n_3}u_3 \\ &\quad \times z^{|u_3|-|u_1|-|u_2|}(\log z)^{n_1+n_2+n_3} \end{aligned} \quad (4.9)$$

for homogeneous $u_1 \in M^1$, $u_2 \in M^2$ and $u_3 \in M^3$. Then the left-hand side of (4.3) is an element in $\mathbb{C}[z, z^{-1}]z^{-h_1-h_2+h_3}[\log z]$. Therefore $\langle I_x(u_2, z)u_1, - \rangle = \langle x, z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle$ gives an element of the space

$$\text{Hom}_{\mathbb{C}}(M_3, \mathbb{C})[[z, z^{-1}]]z^{-h_1-h_2+h_3}[\log z], \quad (4.10)$$

which shows $I_x(u_2, z) \in \text{Hom}_{\mathbb{C}}(M_1, D(M_3))[[z, z^{-1}]]z^{-h_1-h_2+h_3}[\log z]$. Therefore we can write

$$I_x(u_2, z) = \sum_{n=0}^d \sum_{\alpha \in \mathbb{Z}+h_1+h_2-h_3} (u_2)_{(\alpha)}^n z^{-\alpha-1}(\log z)^n. \quad (4.11)$$

For fixed $u_1 \in M_{(h_1+\ell_1)}^1$ and $u_2 \in M_{(h_2+\ell_2)}^2$ with nonnegative integers ℓ_1 and ℓ_2 , we have by (4.9)

$$\langle I_x(u_2, z)u_1, u_3 \rangle = \sum_{n=0}^{k_1+k_2+k_3} \sum_{\ell=0}^{\infty} c_{\ell}^n z^{h_3-h_1-h_2-\ell_1-\ell_2+\ell}(\log z)^n \quad (4.12)$$

where c_{ℓ}^n are complex numbers. The (4.12) implies that $(u_2)_{(\alpha)}^n u_1 = 0$ for $\alpha > h_3 - h_1 - h_2 + \ell_1 + \ell_2 - 1$. Hence $I_x(-, z)$ satisfies the truncation condition.

In order to prove L_{-1} -derivative property, we first note that

$$\begin{aligned} &\langle x, j_0(\omega \otimes z(dz)^{-1})z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\ &\quad + \langle x, z^{-L_0}u_1 \otimes j_1(\omega \otimes z(dz)^{-1})z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\ &\quad + \langle x, z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes j_{\infty}(\omega \otimes z(dz)^{-1})z^{L_0}u_3 \rangle = 0. \end{aligned} \quad (4.13)$$

The left-hand side of (4.13) turns to be

$$\begin{aligned} &\langle x, L_0 z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\ &\quad + \langle x, z^{-L_0}u_1 \otimes (L_0 + L_{-1})z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\ &\quad - \langle x, z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes L_0 z^{L_0}u_3 \rangle. \end{aligned} \quad (4.14)$$

From now on each term in (4.14) is simplified. Let us consider the second term of (4.14). Since $[L_{-1}, L_0] = -L_{-1}$, we have

$$\langle x, z^{-L_0}u_1 \otimes L_{-1}z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle = z \langle I_x(L_{-1}u_2, z)u_1, u_3 \rangle, \quad (4.15)$$

which shows

$$\begin{aligned} z \langle I_x(u_2, z)u_1, u_3 \rangle &= - \langle x, L_0 z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &\quad - \langle x, z^{-L_0} u_1 \otimes L_0 z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &\quad + \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes L_0 z^{L_0} u_3 \rangle. \end{aligned} \quad (4.16)$$

The first term of (4.16) can be calculated to be

$$\begin{aligned} &\langle x, L_0 z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &= \sum_{n_1=0}^{k_1-1} \sum_{n_2=0}^{k_2} \sum_{n_3=0}^{k_3} \frac{(-1)^{n_1+n_2}}{n_1! n_2! n_3!} z^{-|u_1|-|u_2|+|u_3|} (\log z)^{n_1+n_2+n_3} \\ &\quad \times \langle x, (L_0 - |u_1|)^{n_1+1} u_1 \otimes (L_0 - |u_2|)^{n_2} u_2 \otimes (L_0 - |u_3|)^{n_3} u_3 \rangle \\ &\quad + |u_1| \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle. \end{aligned} \quad (4.17)$$

Similarly, the second term of (4.16) becomes

$$\begin{aligned} &\langle x, z^{-L_0} u_1 \otimes L_0 z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &= \sum_{n_1=0}^{k_1} \sum_{n_2=0}^{k_2-1} \sum_{n_3=0}^{k_3} \frac{(-1)^{n_1+n_2}}{n_1! n_2! n_3!} z^{-|u_1|-|u_2|+|u_3|} (\log z)^{n_1+n_2+n_3} \\ &\quad \times \langle x, (L_0 - |u_1|)^{n_1} u_1 \otimes (L_0 - |u_2|)^{n_2+1} u_2 \otimes (L_0 - |u_3|)^{n_3} u_3 \rangle \\ &\quad + |u_2| \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle. \end{aligned} \quad (4.18)$$

Finally, the third term is

$$\begin{aligned} &\langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes L_0 z^{L_0} u_3 \rangle \\ &= \sum_{n_1=0}^{k_1} \sum_{n_2=0}^{k_2} \sum_{n_3=0}^{k_3-1} \frac{(-1)^{n_1+n_2}}{n_1! n_2! n_3!} z^{-|u_1|-|u_2|+|u_3|} (\log z)^{n_1+n_2+n_3} \\ &\quad \times \langle x, (L_0 - |u_1|)^{n_1} u_1 \otimes (L_0 - |u_2|)^{n_2} u_2 \otimes (L_0 - |u_3|)^{n_3+1} u_3 \rangle \\ &\quad - |u_3| \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle. \end{aligned} \quad (4.19)$$

In all of the calculations given above we have used the fact that each M^i is an object in \mathcal{C}_{k_i} . By using (4.16)–(4.19) we obtain

$$\begin{aligned} &\frac{d}{dz} \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &= -z^{-1} \langle x, L_0 z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &\quad - z^{-1} \langle x, z^{-L_0} u_1 \otimes (L_0) z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle \\ &\quad + z^{-1} \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes L_0 z^{L_0} u_3 \rangle, \end{aligned} \quad (4.20)$$

which shows

$$\frac{d}{dz} \langle x, z^{-L_0} u_1 \otimes z^{-L_0} u_2 \otimes z^{L_0} u_3 \rangle = \frac{d}{dz} \langle I_x(u_2, z)u_1, u_3 \rangle. \quad (4.21)$$

Hence we have proved the L_{-1} -derivative property

$$I_x(L_{-1}u_2, z)u_1 = \frac{d}{dz}I_x(u_2, z)u_1. \quad (4.22)$$

Finally we will show (3.5). Since x is a conformal block, for any $p, q \in \mathbb{Z}$ and $a \in V_k$, we have

$$\begin{aligned} & \langle x, j_0(a \otimes z^p(z-1)^q(dz)^{1-k})z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\ & + \langle x, z^{-L_0}u_1 \otimes j_1(a \otimes z^p(z-1)^q(dz)^{1-k})z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\ & + \langle x, z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes j_\infty(a \otimes z^p(z-1)^q(dz)^{1-k})z^{L_0}u_3 \rangle = 0. \end{aligned} \quad (4.23)$$

By (4.5)–(4.7), (2.6) and $[L_0, a_{(n)}] = (k-n-1)a_{(n)}$ we have

$$\begin{aligned} & \sum_{i=1}^{\infty} (-1)^{q+i} \binom{q}{i} z^{-p-i+k-1} \langle I_x(u_2, z) a_{(p+i)} u_1, u_3 \rangle \\ & + \sum_{i=1}^{\infty} \binom{p}{i} z^{-q-i+k-1} \langle I_x(a_{(q+i)} u_2, z) u_1, u_3 \rangle \\ & - \sum_{i=0}^{\infty} (-1)^i \binom{q}{i} z^{-p-q+i+k-1} \langle a_{(p+q-i)} I_x(u_2, z) u_1, u_3 \rangle = 0. \end{aligned} \quad (4.24)$$

Recall that $I_x(u_2, z)u_1 = \sum_{n=0}^d \sum_{\alpha \in \mathbb{C}} (u_2)_{(\alpha)}^n u_1 z^{-\alpha-1} (\log z)^n$. Then taking the coefficient of $z^{-\alpha-p-q+k-2} (\log z)^n$ in (4.24) gives (3.5).

(Step 3) We will show that $F = x$ for any $x \in C^*(M_A, p_A)$ and that $I_F(-, z) = I(-, z)$ for $I(-, z) \in I\left(\begin{smallmatrix} D(M^3) \\ M^2 \ M^1 \end{smallmatrix}\right)$.

Suppose that $I_x(u_2, z) = \sum_{n=0}^d \sum_{\alpha \in \mathbb{C}} (u_2)_{(\alpha)}^n z^{-\alpha-1} (\log z)^n$. By (4.9), we have

$$\begin{aligned} \langle F, u_1 \otimes u_2 \otimes u_3 \rangle &= \langle I_x(u_2, 1) u_1, u_3 \rangle \\ &= \langle (u_2)_{(|u_1|+|u_2|-|u_3|-1)}^0 u_1, u_3 \rangle \\ &= \langle x, u_1 \otimes u_2 \otimes u_3 \rangle \end{aligned} \quad (4.25)$$

for any homogeneous $u_i \in M^i$ ($i = 1, 2, 3$), which implies $F = x$.

Conversely, we see that

$$\begin{aligned}
& \langle I_F(u_2, z)u_1, u_3 \rangle \\
&= \langle F, z^{-L_0}u_1 \otimes z^{-L_0}u_2 \otimes z^{L_0}u_3 \rangle \\
&= \sum_{n_1=0}^{k_1} \sum_{n_2=0}^{k_2} \sum_{n_3=0}^{k_3} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} \\
&\quad \times \langle F, (L_0 - |u_1|)^{n_1}u_1 \otimes (L_0 - |u_2|)^{n_2}u_2 \otimes (L_0 - |u_3|)^{k_3}u_3 \rangle \\
&\quad \times z^{-|u_1|-|u_2|+|u_3|}(\log z)^{n_1+n_2+n_3} \\
&= \sum_{n_1=0}^{k_1} \sum_{n_2=0}^{k_2} \sum_{n_3=0}^{k_3} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} \\
&\quad \times \langle ((L_0 - |u_2|)^{n_2}u_2)_{(\alpha)}^0 (L_0 - |u_1|)^{n_1}u_1, (L_0 - |u_3|)^{n_3}u_3 \rangle \\
&\quad \times z^{-|u_1|-|u_2|+|u_3|}(\log z)^{n_1+n_2+n_3} \\
&= \sum_{n_1=0}^{k_1} \sum_{n_2=0}^{k_2} \sum_{n_3=0}^{k_3} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} \\
&\quad \times \langle (L_0 - |u_3|)^{n_3} ((L_0 - |u_2|)^{n_2}u_2)_{(\alpha)}^0 (L_0 - |u_1|)^{n_1}u_1, u_3 \rangle \\
&\quad \times z^{-|u_1|-|u_2|+|u_3|}(\log z)^{n_1+n_2+n_3}
\end{aligned} \tag{4.26}$$

where $\alpha = |u_1| + |u_2| - |u_3| - 1$. On the other hand, by Lemma 3.2.1, we have

$$\begin{aligned}
& \sum_{n_1+n_2+n_3=k} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} (L_0 - |u_3|)^{n_3} ((L_0 - |u_2|)^{n_2}u_2)_{(\alpha)}^0 (L_0 - |u_1|)^{n_1}u_1 \\
&= \sum_{n_1+n_2+n_3=k} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} \sum_{\ell=0}^{n_3} \binom{n_3}{\ell} \ell! (x_1 + x_2)^{n_3-\ell} x_1^{n_1} x_2^{n_2} (u_2)_{(\alpha)}^\ell u_1 \\
&= \sum_{n_1+n_2+n_3=k} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} \sum_{\ell=0}^{n_3} \binom{n_3}{\ell} \ell! (x_1 + x_2)^{n_3-\ell} x_1^{n_1} x_2^{n_2} (u_2)_{(\alpha)}^\ell u_1 \\
&= \sum_{\ell=0}^k \sum_{n_3=\ell}^k \left(\sum_{n_1+n_2=k-n_3} \frac{(-1)^{n_1+n_2}}{n_1!n_2!(n_3-\ell)!} (x_1 + x_2)^{n_3-\ell} x_1^{n_1} x_2^{n_2} (u_2)_{(\alpha)}^\ell u_1 \right) \\
&= \sum_{\ell=0}^k \sum_{n_3=0}^{k-\ell} \left(\sum_{n_1+n_2=k-\ell-n_3} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} (x_1 + x_2)^{n_3} x_1^{n_1} x_2^{n_2} (u_2)_{(\alpha)}^\ell u_1 \right) \\
&= \sum_{\ell=0}^k \left(\sum_{n_1+n_2+n_3=k-\ell} \frac{(-1)^{n_1+n_2}}{n_1!n_2!n_3!} (x_1 + x_2)^{n_3} x_1^{n_1} x_2^{n_2} (u_2)_{(\alpha)}^\ell u_1 \right) \\
&= \sum_{\ell=0}^k \frac{1}{(k-\ell)!} (-x_1 - x_2 + x_1 + x_2)^{k-\ell} (u_2)_{(\alpha)}^\ell u_1 \\
&= (u_2)_{(\alpha)}^k u_1.
\end{aligned} \tag{4.27}$$

Therefore, by combining (4.26) and (4.27) we obtain

$$\langle I_{F_I}(u_2, z)u_1, u_3 \rangle = \langle I(u_2, z)u_1, u_3 \rangle \quad (4.28)$$

for homogeneous $u_i \in M^i$ ($i = 1, 2, 3$). The theorem is proved.

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